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Persistence and global stability for discrete models of nonautonomous Lotka–Volterra type

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Abstract

Consider the following discrete models of nonautonomous Lotka–Volterra type:

$$\begin{cases} N_i(p+1) = N_i(p)\exp\Big\{c_i(p) - a_i(p)N_i(p) \\ \quad - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l(p)N_j(p-k_l)\Big\}, \quad 1 \leqslant i \leqslant n,\ p = 0, 1, 2, \ldots, \\ N_i(p) = N_{ip} \geqslant 0, \quad p \leqslant 0, \quad \text{and} \quad N_{i0} > 0, \quad 1 \leqslant i \leqslant n, \end{cases}$$

where each $c_i(p)$, $a_i(p)$ and $a_{ij}^l(p)$ are bounded for $p \geqslant 0$ and

$$\begin{cases} \inf_{p\geqslant 0} a_i(p) > 0, \quad a_{ii}^0(p) \equiv 0, \quad 1 \leqslant i \leqslant n, \\ a_{ij}^l(p) \geqslant 0, \quad 1 \leqslant i \leqslant j \leqslant n,\ 0 \leqslant l \leqslant m, \\ k_0 = 0, \quad \text{integers } k_l \geqslant 0, \quad 1 \leqslant l \leqslant m. \end{cases}$$

In this paper, to the above discrete system, we apply the techniques offered by Ahmad and Lazer (Nonlinear Anal. 40 (2000) 37–49), and establish similar conditions of the persistence and global asymptotic stability of the system.

Keywords: Persistence; Global asymptotic stability; Discrete model of nonautonomous Lotka–Volterra type

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1. Introduction

Recently, by using the averaged conditions, Ahmad and Lazer [2] (see also Ahmad and Lazer [1]), have established the strong improvement of the results in Gopalsamy [6,7], and Tineo and Alvarez [13] for the following nonautonomous, competitive, Lotka–Volterra type differential system:

$$\begin{cases} \frac{dx_i(t)}{dt} = x_i(t)\left\{c_i(t) - \sum_{j=1}^n b_{ij}(t)x_j(t)\right\}, & t \geqslant t_0,\ 1 \leqslant i \leqslant n, \\ x_i(t_0) = \phi_i(t_0) > 0, \quad 1 \leqslant i \leqslant n, \end{cases} \tag{1.1}$$

where it is assumed that each $c_i(t)$ and $b_{ij}(t)$ are bounded continuous functions on $[t_0, +\infty)$, and

$$\begin{cases} b_{ij}(t) \geqslant 0, \quad 1 \leqslant i, j \leqslant n, \\ \inf_{t \geqslant t_0} c_i(t) > 0 \quad \text{and} \quad \inf_{t \geqslant t_0} b_{ii}(t) > 0, \quad 1 \leqslant i \leqslant n. \end{cases} \tag{1.2}$$

System (1.1)–(1.2) is a model for n competing species.

The *lower and upper averages of* $g(t)$, *denoted by* $m[g]$ *and* $M[g]$, *respectively*, are defined by

$$m[g] = \lim_{s\to\infty} \inf\left\{ \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} g(s)\,ds \mid t_0 \leqslant t_1 < t_2 \text{ and } t_2 - t_1 \geqslant s \right\} \quad \text{and}$$

$$M[g] = \lim_{s\to\infty} \sup\left\{ \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} g(s)\,ds \mid t_0 \leqslant t_1 < t_2 \text{ and } t_2 - t_1 \geqslant s \right\}. \tag{1.3}$$

Note that

$$\inf_{t \geqslant t_0} g(t) \leqslant m[g] \leqslant M[g] \leqslant \sup_{t \geqslant t_0} g(t), \tag{1.4}$$

and for a periodic or an almost-periodic function g, it holds that $m[g] = M[g]$.

Ahmad and Lazer [2] have established the following theorem.

Theorem 1.1. *For the system* (1.1)–(1.2), *assume the following averaged conditions*:

$$m[c_i] > \sum_{j \neq i} \left(\sup_{t \geqslant t_0} b_{ij}(t)\right) M[c_j] \Big/ \left(\inf_{t \geqslant t_0} b_{jj}(t)\right), \quad 1 \leqslant i \leqslant n. \tag{1.5}$$

Then the followings are true:

(I) *The system is persistent for solutions, that is,*

$$0 < \inf_{t \geqslant t_0} x_i(t) \leqslant \sup_{t \geqslant t_0} x_i(t) < +\infty, \quad 1 \leqslant i \leqslant n. \tag{1.6}$$

(II) *For any two solutions* $x_i(t)$ *and* $y_i(t)$, $1 \leqslant i \leqslant n$, *it holds*

$$\lim_{t\to\infty}\bigl(x_i(t)-y_i(t)\bigr)=0, \quad 1\leqslant i\leqslant n. \tag{1.7}$$

Consider the following discrete system of nonautonomous Lotka–Volterra type:

$$\begin{cases} N_i(p+1)=N_i(p)\exp\bigl\{c_i(p)-a_i(p)N_i(p) \\ \quad -\sum_{j=1}^n\sum_{l=0}^m a_{ij}^l(p)N_j(p-k_l)\bigr\}, \quad p=0,1,2,\ldots, \\ N_i(p)=N_{ip}\geqslant 0, \quad p\leqslant 0, \quad \text{and} \quad N_{i0}>0, \quad 1\leqslant i\leqslant n, \end{cases} \tag{1.8}$$

where we assume that each $c_i(p)$, $a_i(p)$ and $a_{ij}^l(p)$ are bounded for $p\geqslant 0$ and

$$\begin{cases} \inf_{p\geqslant 0} a_i(p)>0, \quad a_{ii}^0(p)\equiv 0, \quad 1\leqslant i\leqslant n, \\ a_{ij}^l(p)\geqslant 0, \quad 1\leqslant i\leqslant j\leqslant n,\ 0\leqslant l\leqslant m, \\ k_0=0, \quad \text{integers } k_l\geqslant 0, \quad 1\leqslant l\leqslant m. \end{cases} \tag{1.9}$$

For autonomous cases in Eq. (1.8), there are several literatures. If for $n=2$ and $m=0$, system is a prey–predator system or the two species are competitive, then a theorem in Hofbauer et al. [4] show that the existence of positive equilibrium in the system guarantee its permanence. But if the system is cooperative, Lu and Wang [8] show that it can not be permanent in any case. Lu and Wang [8] also give sufficient conditions for permanence for no delay case $m=0$; later, Saito et al. [11,12] generalized them and established the necessary and sufficient conditions for permanence in the case $n=2$ and any $m\geqslant 0$.

On the other hand, in the case of prey-predator and competitive system for $n=2$ and $m\geqslant 0$, Wang and Lu [14] and Wang et al. [15] found fined further conditions to ensure that the discrete system is globally asymptotically stable. But for the cases $n\geqslant 2$ and $m\geqslant 0$, it is still a remained problem to establish sufficient conditions for the permanence of the system (1.1).

In this paper, we apply the techniques offered by Ahmad and Lazer [2] to the discrete system (1.8) of nonautonomous Lotka–Volterra type, and establish similar conditions of the persistence and global asymptotic stability of the system.

For a given sequence $\{g(p)\}_{p=0}^{\infty}$, we set

$$\begin{aligned} g_M &= \sup\bigl\{g(p)\mid p=0,1,2,\ldots\bigr\}, \\ g_L &= \inf\bigl\{g(p)\mid p=0,1,2,\ldots\bigr\}, \end{aligned} \tag{1.10}$$

and for integers $0\leqslant p_1<p_2$, we set

$$A[g,p_1,p_2]=\frac{1}{p_2-p_1}\sum_{p=p_1}^{p_2-1} g(p). \tag{1.11}$$

The *lower and upper averages of* $g(p)$, *denoted by* $m[g]$ *and* $M[g]$, *respectively*, are defined by

$$m[g] = \lim_{q\to\infty} \inf\{A[g, p_1, p_2] \mid p_2 - p_1 \geqslant q\} \quad \text{and}$$
$$M[g] = \lim_{q\to\infty} \sup\{A[g, p_1, p_2] \mid p_2 - p_1 \geqslant q\}. \tag{1.12}$$

Since the set $\{A[g, p_1, p_2] \mid p_2 - p_1 \geqslant q\}$ gets smaller as q increases, the limits exist; and since

$$g_L \leqslant A[g, p_1, p_2] \leqslant g_M,$$

it follows that

$$g_L \leqslant m[g] \leqslant M[g] \leqslant g_M. \tag{1.13}$$

Let

$$a^l_{ijL} = a^{l-}_{ijL} + a^{l+}_{ijL}, \quad a^{l-}_{ijL} \leqslant 0 \leqslant a^{l+}_{ijL},$$
$$a^l_{ijM} = a^{l-}_{ijM} + a^{l+}_{ijM}, \quad a^{l-}_{ijM} \leqslant 0 \leqslant a^{l+}_{ijM},$$
$$b_{ijL} = \sum_{l=0}^m a^l_{ijL}, \qquad b^-_{ijL} = \sum_{l=0}^m a^{l-}_{ijL}, \qquad b_{ijM} = \sum_{l=0}^m a^l_{ijM}, \quad \text{and}$$
$$b^+_{ijM} = \sum_{l=0}^m a^{l+}_{ijM}, \quad 1 \leqslant i, j \leqslant n. \tag{1.14}$$

For the discrete system (1.8)–(1.9), we consider an averaged condition as follows:

For any $N_i \geqslant 0$, $1 \leqslant i \leqslant n$, such that

$$M[c_i] \geqslant \left(a_{iL} + \sum_{l=0}^m a^{l+}_{iiL}\right) N_i + \sum_{j=1}^n b^-_{ijL} N_j, \quad 1 \leqslant i \leqslant n, \tag{1.15}$$

it holds that

$$m[c_i] > \sum_{j\neq i} b^+_{ijM} N_j, \quad 1 \leqslant i \leqslant n. \tag{1.16}$$

In particular, for the system (1.8)–(1.9), we see that if $a_i(p) = b_{ii}(p) > 0$, $a^0_{ij}(p) = b_{ij}(p) > 0$, $j \neq i$, and $a^l_{ij}(p) \equiv 0$, $1 \leqslant i, j \leqslant n$, $1 \leqslant l \leqslant m$, then the condition (1.15)–(1.16) becomes the following:

For any $N_i \geqslant 0$, $1 \leqslant i \leqslant n$, such that

$$M[c_i] \geqslant b_{iiL} N_i, \quad 1 \leqslant i \leqslant n, \tag{1.17}$$

it holds that

$$m[c_i] > \sum_{j\neq i} b_{ijM} N_j, \quad 1 \leqslant i \leqslant n. \tag{1.18}$$

In particular, for $N_i = M[c_i]/b_{iiL}$, $1 \leqslant i \leqslant n$, Eq. (1.18) becomes the similar condition to Eq. (1.5), and this condition implies the condition (1.17)–(1.18). Thus, the condition (1.17)–(1.18) for the discrete system (1.8)–(1.9) corresponds to Eq. (1.5) for the system (1.1)–(1.2).

Let

$$\begin{aligned}
&A_L = \operatorname{diag}(a_{1L}, a_{2L}, \ldots, a_{nL}), \quad B_L^- = \left[b_{ijL}^-\right], \quad B_M^+ = \left[b_{ijM}^+\right],\\
&D_L^+ = \operatorname{diag}\left(b_{11L}^+, b_{22L}^+, \ldots, b_{nnL}^+\right), \quad \text{and}\\
&D_M^+ = \operatorname{diag}\left(b_{11M}^+, b_{22M}^+, \ldots, b_{nnM}^+\right)\\
&\text{are } n \times n \text{ matrices, and}\\
&\underline{\boldsymbol{c}} = \left[m[c_i]\right] \quad \text{and} \quad \bar{\boldsymbol{c}} = \left[M[c_i]\right]\\
&\text{are } n\text{-dimensional vectors.}
\end{aligned} \tag{1.19}$$

Then, the condition (1.15)–(1.16) is equivalent to the following:

For any n-dimensional vector $\boldsymbol{N} = [N_i] \geqslant \boldsymbol{0}$ such that

$$\bar{\boldsymbol{c}} \geqslant \left(A_L + D_L^+ + B_L^-\right)\boldsymbol{N}, \tag{1.20}$$

it holds that

$$\underline{\boldsymbol{c}} > \left(B_M^+ - D_M^+\right)\boldsymbol{N}. \tag{1.21}$$

Note that $A_L + B_L^-$ is an M-matrix.

Assume that

$$\left(A_L + D_L^+ + B_L^-\right)^{-1}\bar{\boldsymbol{c}} > \boldsymbol{0} \quad \text{and} \quad \underline{\boldsymbol{c}} > \left(B_M^+ - D_M^+\right)\left(A_L + D_L^+ + B_L^-\right)^{-1}\bar{\boldsymbol{c}}. \tag{1.22}$$

Then, $(A_L + D_L^+ + B_L^-)^{-1} \geqslant \boldsymbol{0}$ and Eq. (1.20) implies that

$$\left(A_L + D_L^+ + B_L^-\right)^{-1}\bar{\boldsymbol{c}} \geqslant \boldsymbol{N},$$

and from $\underline{\boldsymbol{c}} > (B_M^+ - D_M^+)(A_L + D_L^+ + B_L^-)^{-1}\bar{\boldsymbol{c}}$ and $B_M^+ - D_M^+ \geqslant \boldsymbol{0}$, we have that

$$\underline{\boldsymbol{c}} > \left(B_M^+ - D_M^+\right)\left(A_L + D_L^+ + B_L^-\right)^{-1}\bar{\boldsymbol{c}} \geqslant \left(B_M^+ - D_M^+\right)\boldsymbol{N}$$

which implies (1.21).

Thus, the extended averaged condition (1.15)–(1.16) is satisfied.

Let

$$\begin{aligned}
&\tilde{c}_{iM} = c_{iM} - \sum_{j=1}^{i-1} b_{ijL}^- \overline{N}_j, \quad \overline{N}_i = \begin{cases} \tilde{c}_{iM}/a_{iM}, & \tilde{c}_{iM} \leqslant 1,\\ \exp(\tilde{c}_{iM} - 1)/a_{iM}, & \tilde{c}_{iM} > 1,\end{cases}\\
&\tilde{a}_{iL} = \min\left(a_{iL}, \frac{2}{\overline{N}_i} - a_{iM}\right), \quad 1 \leqslant i \leqslant n, \quad \text{and}\\
&\tilde{A}_L = \operatorname{diag}(\tilde{a}_{1L}, \tilde{a}_{2L}, \ldots, \tilde{a}_{nL}).
\end{aligned} \tag{1.23}$$

Note that if $A_L + D_L^+ + B_L^-$ is an M-matrix and $(A_L + D_L^+ + B_L^-)^{-1}\bar{\boldsymbol{c}} > \mathbf{0}$, then for n-dimensional vectors $\overline{N} = [\overline{N}_i]$ and $\boldsymbol{c}_M = [c_{iM}]$, we have that

$$\overline{N} \geqslant \left(A_L + B_L^-\right)^{-1}\boldsymbol{c}_M \geqslant \left(A_L + D_L^+ + B_L^-\right)^{-1}\bar{\boldsymbol{c}} > \mathbf{0}.$$

We shall establish the following extension of the Ahmad and Lazer's results in [2] to the discrete system (1.8)–(1.9).

Theorem 1.2. *For the system* (1.8)–(1.9)*, if the condition* (1.22) *is satisfied, then all solutions of the system are bounded above.*

(I) *If there exists a nonempty subset $Q \in \{1, 2, \ldots, n\}$ such that*

$$c_{iL} - \sum_{j \notin Q} b_{ijM}^+ \overline{N}_j > 0, \quad \text{for any } i \in Q, \tag{1.24}$$

then the system (1.8)–(1.9) *is persistent for solutions, that is,*

$$0 < \liminf_{p \geqslant 0} N_i(p) \leqslant \limsup_{p \geqslant 0} N_i(p) < +\infty, \quad 1 \leqslant i \leqslant n. \tag{1.25}$$

(II) *Moreover, if*

$$\tilde{A}_L - \left(B_M^+ - B_L^-\right) \quad \text{is an } M\text{-matrix}, \tag{1.26}$$

then for any two solutions $\{M_i(p)\}_{p=0}^{\infty}$ and $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leqslant i \leqslant n$, it holds that

$$\lim_{p \to \infty} \left(M_i(p) - N_i(p)\right) = 0, \quad 1 \leqslant i \leqslant n. \tag{1.27}$$

The organization of this paper is as follows. In Section 2, using the similar techniques in Ahmad and Lazer [2], we prove that Eqs. (1.22) and (1.24) $\Rightarrow$ Eq. (1.25), and Eqs. (1.22), (1.24) and (1.26) $\Rightarrow$ Eq. (1.27).

2. Conditions of persistence and global asymptotic stability

Consider the persistence and the global asymptotic stability of the discrete system (1.8)–(1.9) of nonautonomous Lotka–Volterra type.

Lemma 2.1. *For the system* (1.8)–(1.9) *and* $1 \leqslant i \leqslant n$,

$$\begin{aligned} N_i(p) = N_i(0) \exp\Biggl(\sum_{q=0}^{p-1} \Biggl\{ & c_i(q) - a_i(q) x_i(q) \\ & - \sum_{j=1}^{n} \sum_{l=0}^{m} a_{ij}^l(q) N_j(q - k_l) \Biggr\} \Biggr), \quad p \geqslant 0, \end{aligned} \tag{2.1}$$

and every solutions $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leqslant i \leqslant n$ *exist, and remain positive for all* $p = 0, 1, 2, \ldots$.

Proof. Consider the following differential equations with piecewise constant delays:

$$\begin{aligned}
&\frac{dx_i(t)}{dt} = x_i(t)\left\{c_i\big([t]\big) - a_i\big([t]\big)x_i\big([t]\big) - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l\big([t]\big)x_j\big([t-k_l]\big)\right\},\\
&\quad t \geqslant 0,\ 1 \leqslant i \leqslant n,\\
&x_i(t) = \phi_i(t) \geqslant 0, \quad t \leqslant 0, \quad \text{and} \quad \phi_i(0) > 0, \quad 1 \leqslant i \leqslant n,
\end{aligned}$$

where $[t]$ denotes the maximal integer less than or equal to t and each $\phi_i(t)$ is a piecewise constant function such that

$$\phi_i(t) = N_{ip}, \quad [t] = p \leqslant 0.$$

Then we easily see that by Eq. (1.8), $N_i(p) = x_i(p)$, for $p = 0, 1, \ldots$. We have that for any $t \geqslant 0$,

$$\begin{aligned}
\frac{d}{dt}\left\{\frac{1}{x_i(t)}\exp\left(\int_0^t \left\{c_i\big([s]\big) - a_i\big([s]\big)x_i\big([s]\big)\right.\right.\right.\\
\left.\left.\left. - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l\big([s]\big)x_j\big([s-k_l]\big)\right\} ds\right)\right\} = 0.
\end{aligned}$$

Thus, integrating both sides with respect to t on $[0, t]$, one obtains that

$$\begin{aligned}
x_i(t) = x_i(0)\exp\left(\int_0^t \left\{c_i\big([s]\big) - a_i\big([s]\big)x_i\big([s]\big)\right.\right.\\
\left.\left. - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l\big([s]\big)x_j\big([s-k_l]\big)\right\} ds\right), \quad t \geqslant 0,
\end{aligned}$$

from which we get the conclusion. □

Similar to Ahmad and Lazer [2, Lemma 2.1], we have the following lemma (cf. Muroya [9, Theorem 3.1]).

Lemma 2.2. *Assume that for Eq.* (1.19), *and* $\boldsymbol{c}_M = (c_{1M}, c_{2M}, \ldots, c_{nM})^T$

$$\big(A_L + B_L^-\big)^{-1}\boldsymbol{c}_M > \mathbf{0}. \tag{2.2}$$

Then, any solution of the system (1.8)–(1.9) *is bounded above, and it holds that*

$$\limsup_{p\to\infty} N_i(p) \leqslant \overline{N}_i, \quad 1 \leqslant i \leqslant n, \tag{2.3}$$

where $\overline{N}_i$, $1 \leqslant i \leqslant n$, *are defined by* (1.23).

Proof. Since $A_L + B_L^-$ is an M-matrix, it is well known that there is a diagonal matrix $\underline{D} = \operatorname{diag}(\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_n)$ such that $\underline{d}_i > 0$, $1 \leqslant i \leqslant n$, and $(A_L + B_L^-)\underline{D}$ is a diagonally dominant matrix.

Thus, we may assume, without loss of generality, that $A_L + B_L^-$ is diagonally dominant, that is,

$$a_{iL} > 0 \quad \text{and} \quad a_{iL} + \sum_{j=1}^{i-1} b_{ijL}^- > 0, \quad 1 \leqslant i \leqslant n.$$

If $N_1(p) > \overline{N}_1 = c_{1M}/a_{1L}$ for some $p \geqslant 0$, then by assumptions, we have

$$N_1(p+1) \leqslant N_1(p)\exp\bigl\{c_{1M} - a_{1L}N_1(p)\bigr\} < N_1(p).$$

Now, let us consider the case that $N_1(p)$ is eventually decreasing and bounded below by $\tilde{N}_1$. Then, $\lim_{p\to\infty} N_1(p)$ exists. Set

$$\beta = \lim_{p\to\infty} N_1(p) - \tilde{N}_1 \geqslant 0.$$

We will show that $\beta = 0$.

Indeed, suppose $\beta > 0$. Let take any positive constant η. Then, there exists $\bar{p}_0 \geqslant 0$ such that

$$\beta \leqslant N_1(p) - \tilde{N}_1 \leqslant \beta + \eta, \quad \text{for } p \geqslant \bar{p}_0,$$

since $N_1(p) - \tilde{N}_1$ eventually decreases to β. Thus, we have

$$\begin{aligned} N_1(p+1) &\leqslant N_1(p)\exp\bigl\{c_{1M} - a_{1L}N_1(p)\bigr\} \\ &\leqslant N_1(p)\exp\{-a_{1L}\beta\}, \quad \text{for } p \geqslant \bar{p}_0. \end{aligned}$$

Therefore, we have

$$N_1(p+1) \leqslant N_1(\bar{p}_0)\exp\left\{-\beta \int_{\bar{p}_0}^{p+1} a_{1L}\,ds\right\},$$

which in turn implies, due to $\int_{\bar{p}_0}^{\infty} a_{1L}\,dt = +\infty$, $\lim_{p\to\infty} N_1(p) = 0$. This contradicts $N_1(p) \geqslant \tilde{N}_1 + \beta > 0$. Hence, $\lim_{p\to\infty} N_1(p) = \tilde{N}_1$.

On the other hand, if $N_1(p) \leqslant \tilde{N}_1 = c_{1M}/a_{1L}$ for some $p \geqslant 0$, then by (1.8),

$$N_1(p+1) \leqslant N_1(p)\exp\bigl\{c_{1M} - a_{1L}N_1(p)\bigr\}.$$

For $a, c > 0$, consider the function $g(x) = xe^{c-ax}$ of $x \in [0, c/a]$. Since $g'(x) = (1-ax)e^{c-ax}$, we can see that for $0 \leqslant x \leqslant c/a$,

$$\begin{cases} g(x) \leqslant g(c/a) = c/a, & \text{if } c \leqslant 1, \\ g(x) \leqslant g(1/a) = e^{c-1}/a, & \text{if } c > 1. \end{cases}$$

Therefore, in this case, by (1.23), we see that $N_1(p+1) \leqslant \overline{N}_1$.

Hence, we get

$$\limsup_{p\to\infty} N_1(p) \leqslant \overline{N}_1.$$

Next, for some $2 \leqslant i \leqslant n$, suppose that for any fixed positive constant ϵ, there exists a constant $\bar{p}_i \geqslant \bar{p}_{i-1}$ such that

$$N_j(p) \leqslant \overline{N}_j + \epsilon, \quad \text{for any } p \geqslant \bar{p}_i,\ 1 \leqslant j \leqslant i-1.$$

If $N_i(p) > \tilde{N}_i + \epsilon$ for some $p \geqslant \bar{p}_i$, then by (1.8) and (1.23),

$$\begin{aligned} N_i(p+1) &\leqslant N_i(p)\exp\left\{c_{iM} - \sum_{j=1}^{i-1} b_{ijL}^-\big(\overline{N}_j + \epsilon\big) - a_{iL}N_i(p)\right\} \\ &\leqslant N_i(p)\exp\left\{-\left(a_{iL} + \sum_{j=1}^{i-1} b_{ijL}^-\right)\epsilon\right\} < N_i(p). \end{aligned}$$

If $N_i(p)$ is eventually decreasing and bounded below by $\tilde{N}_i + \epsilon$. Then, as similar to the above discussions of $i = 1$, we see $\lim_{p\to\infty} N_i(p) = \tilde{N}_i + \epsilon$.

On the other hand, if $N_i(p) \leqslant \tilde{N}_i + \epsilon$, then by (1.8),

$$\begin{aligned} N_i(p+1) \leqslant N_i(p)\exp&\left\{\left(c_{iM} - \sum_{j=1}^{i-1} b_{ijL}^-\overline{N}_j\right) - a_{iL}N_i(p)\right\} \\ &\times \exp\left\{-\left(\sum_{j=1}^{i-1} b_{ijL}^-\right)\epsilon\right\}. \end{aligned}$$

Therefore, similar to the above discussions of $i = 1$, we get that

$$N_i(p+1) \leqslant \overline{N}_i \exp\left\{-\left(\sum_{j=1}^{i-1} b_{ijL}^-\right)\epsilon\right\}.$$

Thus, we have

$$\limsup_{p\to\infty} N_i(p) \leqslant \overline{N}_i \exp\left\{-\left(\sum_{j=1}^{i-1} b_{ijL}^-\right)\epsilon\right\}.$$

Since $\epsilon > 0$ is any positive constant, by inductions of $i = 1, 2, \ldots, n$, we derive that

$$\limsup_{p\to\infty} N_i(p) \leqslant \overline{N}_i, \quad 1 \leqslant i \leqslant n.$$

Hence, we complete the proof. $\square$

Similar to Ahmad and Lazer [2, Lemma 2.2] we have a lemma.

Lemma 2.3. *Assume Eqs.* (2.2) *and* (1.24)*. Then, for solutions* $\{N_i(p)\}_{p=0}^{\infty}$*,* $1 \leqslant i \leqslant n$*, of the system* (1.8)–(1.9)*, there exists a number* $\alpha > 0$ *such that for all* $p \geqslant 0$*,*

$$\sum_{i \in Q} N_i(p) \geqslant \alpha. \tag{2.4}$$

Proof. By assumptions to system (1.8)–(1.9), there exist positive constants $\underline{\gamma}$, $\bar{b}_l$, $0 \leqslant l \leqslant m$, such that for $i \in Q$,

$$c_{iL} - \sum_{j \notin Q} b_{ijM}^{+} \overline{N}_j \geqslant \underline{\gamma} \quad \text{and} \quad a_{iM}, a_{ijM}^{l} \leqslant \bar{b}_l,$$
$$j \in Q,\ 0 \leqslant l \leqslant m.$$

By Eq. (1.9), it follows that there exists a nonnegative integer $\bar{p}_1$ such that for $p \geqslant \bar{p}_1$ and $i \in Q$,

$$\begin{aligned} N_i(p+1) \geqslant N_i(p)\Biggl\{&\Biggl(c_{iL} - \sum_{j \notin Q} \sum_{l=0}^{m} a_{ijM}^{l+} N_j(p-k_l)\Biggr) - a_i(p)N_i(p) \\ &- \sum_{j \in Q} \sum_{l=0}^{m} a_{ij}^{l}(p) N_j(p-k_l)\Biggr\} \\ \geqslant N_i(p)\Biggl\{&\underline{\gamma} - \sum_{l=0}^{m} \bar{b}_l \sum_{j \in Q} N_j(p-k_l)\Biggr\}. \end{aligned}$$

This shows that if

$$V(p) = \sum_{j \in Q} N_j(p),$$

then

$$V(p+1) \geqslant V(p)\Biggl\{\underline{\gamma} - \sum_{l=0}^{m} \bar{b}_l V(p-k_l)\Biggr\}. \tag{2.5}$$

Now, suppose that $\liminf_{p\to\infty} V(p) = 0$. Then, there exists a sequence $\{p_q\}_{q=1}^{\infty}$ such that

$$\ln\Bigl\{V(p_q+1)/V(p_q)\Bigr\} \leqslant 0 \quad \text{and} \quad \lim_{q\to\infty} V(p_q) = 0.$$

Since $V(t) > 0$ and for $V^* = \underline{\gamma}/(\sum_{l=0}^{m} \bar{b}_l) > 0$,

$$\ln\bigl\{V(p+1)/V(p)\bigr\} \geqslant \sum_{l=0}^{m} \bar{b}_l\bigl(V^* - V(p-k_l)\bigr),$$

it holds that for each $q \geqslant 1$, there exists a $l_q \in \{0, 1, 2, \dots, m\}$ such that

$$V(p_q - k_{l_q}) \geqslant V^*.$$

Similar to Eq. (2.1), it follows from Eq. (2.5) that

$$V(p_q) \geqslant \frac{V(p_q - k_{l_q}) \exp(\sum_{p=p_q-k_{l_q}}^{p_q-1} \{\underline{\gamma} - \sum_{l=0}^{m} \bar{b}_l V(p - k_l)\})}{1 + \bar{b}_0 V(p_q - k_{l_q}) \sum_{p=p_q-k_{l_q}}^{p_q-1} \exp(\sum_{r=p_q-k_{l_q}}^{r-1} \{\underline{\gamma} - \sum_{l=0}^{m} \bar{b}_l V(r - k_l)\})}.$$

By Lemma 2.2 and assumptions, there is a positive constant $\overline{V}$ such that $V(p) \leqslant \overline{V}$, $p \geqslant 0$, and for $\bar{k} = \max_{0 \leqslant l \leqslant m} k_l$, we have that

$$V(p_q) \geqslant \beta \equiv V^* \exp\left(\left\{\underline{\gamma} - \sum_{l=0}^{m} \bar{b}_l \overline{V}\right\} \bar{k}\right) \Big/ \left\{1 + \bar{b}_0 V^* \bar{k} \exp(\underline{\gamma} \bar{k})\right\} > 0,$$
$$q \geqslant 1,$$

which is a contradiction.

Therefore,

$$\liminf_{p \to \infty} V(p) > 0,$$

and, hence, Eq. (2.4) holds. □

From Lemma 2.3, we easily obtain the following lemma (see Ahmad and Lazer [2, Lemma 2.3]).

Lemma 2.4. *Assume Eqs.* (2.2) *and* (1.24) *and suppose that Eq.* (1.25) *does not hold. Then, for solutions* $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leqslant i \leqslant n$ *of the system* (1.8)–(1.9), *there exists a maximal nonempty subset* J *of* $\{1, 2, \dots, n\}$ *such that*

$$J \neq \{1, 2, \dots, n\} \tag{2.6}$$

and

$$\inf_{p \geqslant 0} \max\left\{N_j(p) \mid j \in J\right\} = 0. \tag{2.7}$$

The following lemma is a discrete version, similar to Ahmad and Lazer [2, Lemma 2.4].

Lemma 2.5. *Assume Eqs.* (2.2) *and* (1.24) *and suppose that Eq.* (1.25) *does not hold. Let* $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leqslant i \leqslant n$, *and* J *be as in Lemma* 2.4, *and*

$$\min\left\{N_j(0) \mid j \in J\right\} = \delta > 0. \tag{2.8}$$

Then, there exist integer sequences $\{s_q\}_{q=1}^{\infty}$ *and* $\{t_q\}_{q=1}^{\infty}$ *such that for any* $q \geqslant 1$,

$$\begin{cases} 0 \leqslant s_q < t_q, \\ t_q - s_q \geqslant q, \\ \max\{N_j(p) \mid j \in J,\ s_q \leqslant p \leqslant t_q\} \leqslant \delta/q, \end{cases} \tag{2.9}$$

and there exists $j_q \in J$ *such that*

$$\begin{aligned} N_{j_q}(s_q) &= \max\bigl\{N_j(p) \mid j \in J,\ s_q \leqslant p \leqslant t_q\bigr\} \\ &\geqslant \delta/(q+1) \geqslant N_{j_q}(t_q). \end{aligned} \tag{2.10}$$

Proof. Since the functions $c_i(p)$, $a_i(p)$ and $a_{ij}^l(p)$ are bounded for $p \geqslant 0$ and, by Lemma 2.2, each $N_i(p)$ is bounded for $p \geqslant 0$, it follows that for

$$r_i(p) \equiv c_i(p) - a_i(p)N_i(p) - \sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l N_j(p-k_l), \quad i = 1, 2, \ldots, n,$$

there exists a number $\bar{r} > 0$ such that for $i = 1, 2, \ldots, n$,

$$r_i(p) \geqslant -\bar{r}, \quad \text{for any } p \geqslant 0. \tag{2.11}$$

By definition of the set J, it follows that for each integer $q \geqslant 1$, there exists a number $t_q \geqslant 0$ such that

$$\max\bigl\{N_j(t_q) \mid j \in J\bigr\} \leqslant \delta e^{-q\bar{r}}/(q+1). \tag{2.12}$$

By Eq. (2.8) and continuity, there exists a number s_q with $0 \leqslant s_q < t_q$ such that

$$\max\bigl\{N_j(p) \mid j \in J\bigr\} \leqslant \delta/q,$$

for $s_q \leqslant p \leqslant t_q$ and

$$\max\bigl\{N_j(s_q) \mid j \in J\bigr\} = \max\bigl\{N_j(p) \mid j \in J,\ s_q \leqslant p \leqslant t_q\bigr\} \geqslant \delta/(q+1).$$

Let $j_q \in J$ be an integer such that $N_{j_q}(s_q) = \max\{N_j(p) \mid j \in J,\ s_q \leqslant p \leqslant t_q\}$. Then,

$$N_{j_q}(s_q) \geqslant \frac{\delta}{q+1} \geqslant N_{j_q}(t_q), \quad q \geqslant 1.$$

Since for $s_q \leqslant p \leqslant t_q$,

$$\ln\bigl\{N_{j_q}(p+1)/N_{j_q}(p)\bigr\} = r_{j_q}(p),$$

we see that

$$\delta/(q+1) \geqslant N_{j_p}(s_q) = N_{j_q}(t_q)\exp\Biggl\{-\sum_{p=s_q}^{t_q-1} r_{j_q}(p)\Biggr\}.$$

Therefore, from Eqs. (2.11) and (2.12),

$$\delta/(q+1) \leqslant \delta e^{-q\bar{r}} e^{\bar{r}(t_q-s_q)}/(q+1).$$

It follows that

$$e^{\bar{r}(t_q - s_q - q)} \geqslant 1,$$

which implies that $t_q - s_q \geqslant q$. This proves the lemma. □

The following lemma is similar to Ahmad and Lazer [2, Lemma 2.5].

Lemma 2.6. *Assume Eqs.* (2.2) *and* (1.24) *and suppose that Eq.* (1.25) *does not hold. Let* $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leqslant i \leqslant n$, J, *and the sequences* $\{s_q\}_{q=1}^{\infty}$, $\{t_q\}_{q=1}^{\infty}$ *be as in Lemma* 2.5, *and* K *be the subset of* $\{1, 2, \ldots, n\}$ *such that* $J \cap K = \phi$ *and* $J \cup K = \{1, 2, \ldots, n\}$. *Then, there exists a number* $\epsilon > 0$ *such that for all* $q \geqslant 1$ *and all* $k \in K$,

$$N_k(p) \geqslant \epsilon, \quad \textit{for any } p \in [s_q, t_q]. \tag{2.13}$$

Proof. If such a number $\epsilon > 0$ did not exist, there would be an integer $k_* \in K$, a sequence of integers $\{q_r\}_{r=1}^{\infty}$ and a sequence $\{t_{q_r}^*\}_{r=1}^{\infty}$ such that

$$s_{q_r} \leqslant t_{q_r}^* \leqslant t_{q_r} \quad \text{and} \quad \lim_{r \to \infty} N_{k_*}\big(t_{q_r}^*\big) = 0.$$

Since $N_j(p) \leqslant \delta/q_r$, for any p: $s_{q_r} \leqslant p \leqslant t_{q_r}$ and $j \in J$, it follows that if

$$J^* = J \cup \{k_*\},$$

then

$$\inf_{p \geqslant 0} \max\big\{N_i(p) \mid i \in J^*\big\} = 0.$$

But $J \subseteq J^*$ and $J \neq J^*$, so we have a contradiction to the fact that J is a *maximal subset* of $\{1, 2, \ldots, n\}$ with this property. This proves the lemma. □

Continuing the proof that Eqs. (1.22) and (1.24) $\Rightarrow$ Eq. (1.25), we note that if Eq. (1.25) does not hold, then for $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leqslant i \leqslant n$, J, and j_q in Lemma 2.5, there exists an integer $j_* \in J$ such that $j_q = j_*$ for infinitely many integers q. Let $\{q_r\}_{r=1}^{\infty}$ be an increasing sequence of integers such that

$$j_{q_r} = j_*, \quad \text{for any } r \geqslant 1. \tag{2.14}$$

To simplify the notation, let $c_r = s_{q_r}$ and $d_r = t_{q_r}$ for $r \geqslant 1$, so

$$d_r - c_r \geqslant q_r, \quad \text{for any } r \geqslant 1. \tag{2.15}$$

Since, according to Eq. (2.9),

$$\max\big\{N_j(p) \mid j \in J,\ c_r \leqslant p \leqslant d_r\big\} \leqslant \delta/q_r,$$

and for $0 \leqslant l \leqslant m$,

$$\frac{1}{d_r - c_r}\sum_{p=c_r}^{d_r-1} N_j(p-k_l) = \frac{1}{d_r - c_r}\sum_{p=c_r}^{d_r-1} N_j(p) + \frac{1}{d_r - c_r}\sum_{p=c_r-k_l}^{c_r-1} N_j(p) - \frac{1}{d_r - c_r}\sum_{p=d_r-k_l}^{d_r-1} N_j(p), \tag{2.16}$$

we have that for any $j \in J$,

$$\lim_{r\to\infty} \frac{1}{d_r - c_r}\sum_{p=c_r}^{d_r-1} N_j(p-k_l) = 0, \quad 0 \leqslant l \leqslant m. \tag{2.17}$$

Since, according to Eqs. (2.9) and (2.10), $N_{j_*}(c_r) = \delta/(q_r+1) \geqslant N_{j_*}(d_r)$, we have that for any $r \geqslant 1$,

$$\ln\bigl(N_{j_*}(d_r)/N_{j_*}(c_r)\bigr) \leqslant 0. \tag{2.18}$$

The following lemma is a basic result and similar to Ahmad and Lazer [2, Lemma 2.6].

Lemma 2.7. *Assume Eqs.* (2.2) *and* (1.24) *and suppose that Eq.* (1.25) *does not hold. Let J and K be as in Lemmas* 2.5 *and* 2.6*. Then there exists for each $k \in K$, a number $N_k > 0$ such that for any $k \in K$,*

$$M[c_k] \geqslant a_{kL} N_k + \sum_{j\in K} b_{kjL} N_j, \tag{2.19}$$

and there exists a $j_* \in J$ such that

$$m[c_{j_*}] \leqslant \sum_{k\in K} b_{j_*kM} N_k. \tag{2.20}$$

Proof. According to Lemmas 2.1 and 2.2, we have that there exists a positive number R such that

$$0 < N_k(p) \leqslant R, \quad \text{for any } p \geqslant 0 \text{ and } k \in K.$$

Therefore, it holds that

$$0 \leqslant \frac{1}{d_r - c_r}\sum_{p=c_r}^{d_r-1} N_k(p) \leqslant R, \quad \text{for any } k \in K \text{ and } r \geqslant 1.$$

Therefore, without loss of generality, by considering subsequences of $\{c_r\}_{r=1}^{\infty}$ and $\{d_r\}_{r=1}^{\infty}$ if necessary, we may assume that for all $k \in K$,

$$\lim_{r\to\infty} \frac{1}{d_r - c_r}\sum_{p=c_r}^{d_r-1} N_k(p) \equiv N_k > 0 \tag{2.21}$$

exists (see Eqs. (2.3) and (2.13)). Then, by Eq. (2.16) for $k \in K$, it follows that

$$\lim_{r\to\infty} \frac{1}{d_r - c_r} \sum_{p=c_r}^{d_r-1} N_k(p - k_l) = N_k > 0, \quad 0 \leqslant l \leqslant m. \tag{2.22}$$

Since $c_{kL} \leqslant c_k(p) \leqslant c_{kM}$, for any $p \geqslant 0$, by consideration of subsequences of $\{c_r\}_{r=1}^{\infty}$ and $\{d_r\}_{r=1}^{\infty}$ if necessary, we may assume the existence of

$$\lim_{r\to\infty} \frac{1}{d_r - c_r} \sum_{p=c_r}^{d_r-1} c_k(p)$$

for all $k \in K$.

Moreover, since $d_r - c_r \geqslant q_r \to \infty$ as $r \to \infty$, it follows that for all $k \in K$,

$$\lim_{r\to\infty} \frac{1}{d_r - c_r} \sum_{c_r}^{d_r-1} c_k(p) \leqslant M[c_k]. \tag{2.23}$$

Since, by Lemma 2.6 and the above,

$$\epsilon \leqslant N_k(c_r), \quad N_k(d_r) \leqslant R, \quad \text{for any } k \in K \text{ and } q \geqslant 1,$$

we have that for all $k \in K$,

$$\lim_{r\to\infty} \frac{1}{d_r - c_r} \ln \frac{N_k(d_r)}{N_k(c_r)} = 0. \tag{2.24}$$

From Eqs. (1.8) and (1.9), we see that for $k \in K$,

$$\begin{aligned} c_k(p) &= \ln\bigl\{N_k(p+1)/N_k(p)\bigr\} + a_k(p)N_k(p) + \sum_{j=1}^{n}\sum_{l=0}^{m} a_{kj}^l(p)N_j(p-k_l) \\ &\geqslant \ln\bigl\{N_k(p+1)/N_k(p)\bigr\} + a_{kL}N_k(p) + \sum_{j=1}^{n}\sum_{l=0}^{m} a_{kjL}^l N_j(p-k_l). \end{aligned}$$

Therefore, by Eqs. (2.17) and (2.21)–(2.24), for any $k \in K$,

$$\begin{aligned} M[c_k] &\geqslant \lim_{r\to\infty} \frac{1}{d_r - c_r} \sum_{p=c_r}^{d_r-1} c_k(p) \\ &\geqslant \lim_{r\to\infty} \frac{1}{d_r - c_r} \ln \frac{N_k(d_r)}{N_k(c_r)} + a_{kL}N_k + \sum_{j\in K} b_{kjL}N_j \\ &= a_{kL}N_k + \sum_{j\in K} b_{kjL}N_j, \end{aligned}$$

which implies Eq. (2.19).

Finally, we take j_* defined in Eq. (2.14).

Since $c_{j_*L} \leqslant c_{j_*}(p) \leqslant c_{j_*M}$, we may assume that

$$\lim_{r\to\infty} \frac{1}{d_r - c_r} \sum_{p=c_r}^{d_r-1} c_{j_*}(p)$$

exists. Clearly, from Eqs. (1.12) and (2.15),

$$m[c_{j_*}] \leqslant \lim_{r\to\infty} \frac{1}{d_r - c_r} \sum_{p=c_r}^{d_r-1} c_{j_*}(p).$$

From Eqs. (1.8)–(1.9), we have

$$\begin{aligned} c_{j_*}(p) &= \ln\bigl\{N_{j_*}(p+1)/N_{j_*}(p)\bigr\} + a_{j_*}(p)N_{j_*}(p) \\ &\quad + \sum_{k=1}^{n}\sum_{l=0}^{m} a^l_{j_*k}(p)N_k(p-k_l) \\ &\leqslant \ln\bigl\{N_{j_*}(p+1)/N_{j_*}(p)\bigr\} + a_{j_*M}N_{j_*}(p) \\ &\quad + \sum_{k=1}^{n}\sum_{l=0}^{m} a^l_{j_*kM}N_k(p-k_l). \end{aligned}$$

Therefore, from Eqs. (2.17), (2.18) and (2.22),

$$\begin{aligned} m[c_{j_*}] &\leqslant \lim_{r\to\infty} \frac{1}{d_r - c_r} \sum_{p=c_r}^{d_r-1} c_{j_*}(p) \\ &\leqslant \lim_{r\to\infty} \frac{1}{d_r - c_r} \ln\frac{N_{j_*}(d_r)}{N_{j_*}(c_r)} + \sum_{k\in K} b_{j_*kM}N_k \\ &\leqslant \sum_{k\in K} b_{j_*kM}N_k, \end{aligned}$$

which implies Eq. (2.20). This proves the lemma. □

It is now easy to finish the proof that Eqs. (1.22) and (1.24) ⇒ Eq. (1.25) by contradiction. Suppose that Eqs. (1.22) and (1.24) are true but Eq. (1.25) does not hold. Then, by the above lemmas, there exist two subsets J and K of $\{1, 2, \ldots, n\}$, an integer $j_* \in J$ and positive numbers N_k, $k \in K$, such that Eqs. (2.19) and (2.20) hold.

Let $N_j = 0$ for any $j \in J$ and $\boldsymbol{N} = (N_1, N_2, \ldots, N_n)^{\mathrm{T}}$. From Eq. (2.19), we see that

$$M[c_k] \geqslant a_{kL}N_k + \sum_{j\in K} b_{kjL}N_j \geqslant \left(a_{kL} + \sum_{l=0}^{m} a^{l+}_{kkL}\right)N_k + \sum_{j=1}^{n} b^-_{kjL}N_j,$$

for any $k \in K$,

$$M[c_i] \geqslant 0 \geqslant a_{iL} N_i + \sum_{j=1}^{n} b_{ijL}^{-} N_j, \quad \text{for any } i \in J.$$

Then by Eqs. (1.19) and (1.22), it holds that

$$\boldsymbol{N} \leqslant \left(A_L + D_L^{+} + B_L^{-}\right)^{-1} \bar{\boldsymbol{c}}. \tag{2.25}$$

It follows from Eq. (2.20) that

$$m[c_{j_*}] \leqslant \sum_{k \in K} b_{j_* k M} N_k \leqslant \sum_{k \in K} b_{j_* k M}^{+} N_k.$$

However, by conditions (1.22) and (2.25), we see that

$$\underline{\boldsymbol{c}} > \left(B_M^{+} - D_M^{+}\right)\left(A_L + D_L^{+} + B_L^{-}\right)^{-1} \bar{\boldsymbol{c}} \geqslant \left(B_M^{+} - D_M^{+}\right)\boldsymbol{N}.$$

Therefore, by Eq. (1.19), it holds that

$$m[c_{j_*}] > \sum_{k \neq j_*} b_{j_* k M}^{+} N_k = \sum_{k \in K} b_{j_* k M}^{+} N_k.$$

This contradiction proves that Eqs. (1.22) and (1.24) $\Rightarrow$ Eq. (1.25). $\square$

Now, consider the proof that Eqs. (1.22), (1.24) and (1.26) $\Rightarrow$ Eq. (1.27). The proof relies on similar ideas already used by Gopalsamy [5], Tineo and Alvalez [13], Redheffer [10], Ahmad and Lazer [1,2] for nonautonomous, competitive, Lotka–Volterra type differential system (1.1), but contain a little improved version of discrete type (cf. Wang et al. [15]).

Lemma 2.8. *For the system* (1.8)–(1.9), *assume the conditions* (1.22) *and* (1.24), *and suppose that there exist positive constants* $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\eta > 0$ *and a positive integer* p_0 *such that for* $p \geqslant p_0$,

$$\alpha_i \tilde{a}_i(p) - \sum_{j \neq i} \alpha_j \left|a_{ji}^{0}(p)\right| - \sum_{j=1}^{n} \alpha_j \sum_{l=1}^{m} \left|a_{ji}^{l}(p + k_l)\right| \geqslant \eta,$$
$$1 \leqslant i \leqslant n, \tag{2.26}$$

where

$$\tilde{a}_i(p) = \min\left(a_i(p), \frac{2}{\overline{N}_i} - a_i(p)\right), \quad 1 \leqslant i \leqslant n. \tag{2.27}$$

Then any two solutions $\{M_i(p)\}_{p=0}^{\infty}$, $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leqslant i \leqslant n$, *of the system* (1.8)–(1.9) *satisfy the condition*

$$\lim_{p \to \infty} \left(M_i(p) - N_i(p)\right) = 0, \quad 1 \leqslant i \leqslant n. \tag{2.28}$$

Proof. For the positive constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ in Eq. (2.26), consider a Lyapunov-like nonnegative sequence $\{v(p)\}_{p=0}^{\infty}$ such that for $p \geqslant 0$,

$$v(p) = \sum_{i=1}^{n} \alpha_i \left\{ \left| \ln \frac{M_i(p)}{N_i(p)} \right| + \sum_{j=1}^{n} \sum_{l=1}^{m} \sum_{q=p-k_l}^{p-1} \left| a_{ij}^l(q+k_l) \right| \left| M_j(q) - N_j(q) \right| \right\}. \tag{2.29}$$

From Eq. (1.8), one can verify that for $p \geqslant 0$ and $1 \leqslant i \leqslant n$,

$$\begin{aligned} \left| \ln \frac{M_i(p+1)}{N_i(p+1)} \right| \leqslant{} & \left| \ln \frac{M_i(p)}{N_i(p)} - a_i(p)\big(M_i(p) - N_i(p)\big) \right| \\ & + \sum_{j \neq i} \left| a_{ij}^0(p) \right| \left| M_j(p) - N_j(p) \right| \\ & + \sum_{j=1}^{n} \sum_{l=1}^{m} \left| a_{ij}^l(p) \right| \left| M_j(p-k_l) - N_j(p-k_l) \right|. \end{aligned} \tag{2.30}$$

By Eq. (1.22) and Lemma 2.2, for any fixed positive constant ϵ, there is a positive integer $\bar{p}_\epsilon$ such that for $p \geqslant \bar{p}_\epsilon$,

$$M_i(p), N_i(p) \leqslant \overline{N}_i + \epsilon, \quad 1 \leqslant i \leqslant n.$$

Then, for $p \geqslant \bar{p}_\epsilon$, we have that

$$M_i(p) - N_i(p) = e^{\ln M_i(p)} - e^{\ln N_i(p)} = \xi_i(p) \ln \frac{M_i(p)}{N_i(p)} \quad \text{and}$$
$$0 < \xi_i(p) < \max\big\{M_i(p), N_i(p)\big\} \leqslant \overline{N}_i + \epsilon, \quad 1 \leqslant i \leqslant n,$$

which implies that for $1 \leqslant i \leqslant n$,

$$\begin{aligned} & \left| \ln \frac{M_i(p)}{N_i(p)} - a_i(p)\big(M_i(p) - N_i(p)\big) \right| \\ & \quad = \left| \ln \frac{M_i(p)}{N_i(p)} \right| - \left(\frac{1}{\xi_i(p)} - \left| \frac{1}{\xi_i(p)} - a_i(p) \right| \right) \left| M_i(p) - N_i(p) \right|. \end{aligned} \tag{2.31}$$

By Eqs. (1.22) and (1.24) $\Rightarrow$ Eq. (1.25) in Theorem 1.2, $\{M_i(p)\}_{p=0}^{\infty}$ and $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leqslant i \leqslant n$, are bounded above and below by positive constants for $p \geqslant 0$. Therefore, it follows that for any $p \geqslant \max_{0 \leqslant l \leqslant m} k_l$,

$$v(p) = \sum_{i=1}^{n} \alpha_i \left\{ \left| \ln \frac{M_i(p)}{N_i(p)} \right| + \sum_{j=1}^{n} \sum_{l=1}^{m} \sum_{q=p-k_l}^{p-1} \left| a_{ij}^l(q+k_l) \right| \left| M_j(q) - N_j(q) \right| \right\} < +\infty.$$

Using Eqs. (2.29)–(2.31) and (2.26), one can obtain that there exists an $\epsilon > 0$ such that $0 < \eta_\epsilon \leqslant \eta$ and for $p \geqslant \bar{p}_\epsilon$,

$$\begin{aligned} v(p+1) &\leqslant v(p) - \sum_{i=1}^{n} \Biggl\{ \alpha_i \tilde{\tilde{a}}_i(p) - \sum_{j \neq i} \alpha_j \bigl|a_{ji}^0(p)\bigr| \\ &\qquad - \sum_{j=1}^{n} \sum_{l=1}^{m} \alpha_j \bigl|a_{ji}^l(p+k_l)\bigr| \Biggr\} \bigl|M_i(p) - N_i(p)\bigr| \\ &\leqslant v(p) - \eta_\epsilon \sum_{i=1}^{n} \bigl|M_i(p) - N_i(p)\bigr|, \end{aligned}$$

where

$$\begin{aligned} \tilde{\tilde{a}}_i(p) &= \min\left(a_i(p), \frac{2}{\xi_i(p)} - a_i(p) \right), \\ \tilde{a}_i^\epsilon(p) &= \min\left(a_i(p), \frac{2}{\overline{N}_i + \epsilon} - a_i(p) \right), \quad 1 \leqslant i \leqslant n, \end{aligned}$$

and

$$\begin{aligned} \eta \geqslant \eta_\epsilon \equiv \min_{1 \leqslant i \leqslant n} \inf_{p \geqslant \bar{p}_\epsilon} \Biggl\{ &\alpha_i \tilde{a}_i^\epsilon(p) - \sum_{j \neq i} \alpha_j \bigl|a_{ji}^0(p)\bigr| \\ &- \sum_{j=1}^{n} \alpha_j \sum_{l=1}^{m} \bigl|a_{ji}^l(p+k_l)\bigr| \Biggr\} > 0. \end{aligned}$$

Therefore, $\{v(p)\}_{p=\bar{p}_\epsilon}^{\infty}$ is a strictly monotone decreasing sequence, and by (1.25), we obtain $\lim_{p\to\infty} v(p) = 0$, and hence we have that

$$\lim_{p\to\infty} \sum_{i=1}^{n} \alpha_i \left| \ln \frac{M_i(p)}{N_i(p)} \right| = 0,$$

which implies Eq. (2.28). □

If $\{M_i(p)\}_{p=0}^{\infty}$ and $\{N_i(p)\}_{p=0}^{\infty}$, $1 \leqslant i \leqslant n$, are any two solutions of the system (1.8)–(1.9) with $M_i(p) \geqslant 0$, $N_i(p) \geqslant 0$, $p < 0$, and $M_i(0) > 0$, $N_i(0) > 0$, for $1 \leqslant i \leqslant n$, then since we have shown that Eqs. (1.22) and (1.24) $\Rightarrow$ Eq. (1.25), it follows that if Eqs. (1.22) and (1.24) hold, then there exist positive constants δ and R such that $\delta \leqslant M_i(p), N_i(p) \leqslant R$, for $1 \leqslant i \leqslant n$ and $p \geqslant 0$. Therefore, by Lemma 2.8, in order to prove Eqs. (1.22), (1.24) and (1.26) $\Rightarrow$ Eq. (1.27), it is sufficient to show that condition Eq. (1.26) implies the existence of $\alpha_i > 0$, $1 \leqslant i \leqslant n$ such that Eq. (2.26) hold for all $j = 1, 2, \ldots, n$ and $p \geqslant 0$. We can see that Eq. (1.26) implies stronger inequalities in Eq. (2.32), because the following lemma is well known (see, for example, Berman and Plemmons [3]).

Lemma 2.9. *Condition* (1.26) *holds if and only if there exist constants* $\alpha_i > 0$, $1 \leqslant i \leqslant n$, *such that for* $1 \leqslant i \leqslant n$,

$$\alpha_i \tilde{a}_{iL} - \sum_{j \neq i} \alpha_j \left(a_{jiM}^{0+} - a_{jiL}^{0-}\right) - \sum_{j=1}^{n} \alpha_j \sum_{l=1}^{m} \left(a_{jiM}^{l+} - a_{jiL}^{l-}\right) > 0, \quad 1 \leqslant i \leqslant n. \tag{2.32}$$

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